A dual view of the 3d Heisenberg model and the abelian projection.

A. Di Giacomo^a, D. Martelli^b and G. Paffuti^a

a) Dipartimento di Fisica Università and INFN Pisa (Italy)

b) ISAS and INFN Trieste (Italy)

Abstract

The Heisenberg model in 3d is studied from a dual point of view. It is shown that it can have vortex configurations, carrying a conserved charge ($U(1)$ symmetry).

Vortices condense in the disordered (demagnetized) phase. A disorder parameter $\langle \mu \rangle$ is defined, dual to the magnetization $\langle \vec{n} \rangle$, which signals condensation of vortices, i.e. spontaneous breaking of the dual $U(1)$ symmetry. This study sheds light on the procedure known as abelian projection in non abelian gauge theories.

1   Introduction

Order disorder duality[1, 2] plays an increasingly important role in our understanding of gauge theories, specifically of QCD[3] and of its supersymmetric extensions[4].

Duality exists in systems which have spatial configurations with non trivial topology, carrying a conserved topological charge. It consists in the possibility of describing the system in terms of two sets of fields: a set of local fields which is convenient to describe the ordered phase at weak coupling (low temperature), and a dual set which is convenient at strong coupling (high temperature). The non zero vacuum expectation value (vev) of a local field (order parameter) $\langle \vec{\Phi} \rangle$ signals order. At some value of the coupling (temperature) $\langle \vec{\Phi} \rangle \to 0$ and there is a transition to disorder. In the disordered phase topological excitations condense, the symmetry corresponding to conservation of the topological charge is spontaneously broken. In terms of the dual variables the system looks ordered, and the non zero vev of some operator μ, $\langle \mu \rangle$ carrying non zero topological charge is the dual order parameter. When expressed in terms of the original fields μ is a highly non local operator. The dual order parameter is usually called a disorder parameter.

The prototype system showing duality is the 2d Ising model. The field is a two valued variable $\sigma(n) = \pm 1$, defined on the sites of a 2d square lattice. At low temperature there is order and $\langle \sigma \rangle \neq 0$, breaking the symmetry $\sigma \leftrightarrow -\sigma$. A transition point exists where $\langle \sigma \rangle \to 0$; at higher temperatures the system looks disordered. However the same system can be described in terms of a dual variable $\sigma^*(n^*) = \pm 1$, again on a square lattice which roughly speaking associates a site to each spatial link of the original lattice with the rule that $\sigma^* = -1$ if the values of σ at the ends of the link are equal and $\sigma^* = 1$ if they are different.

Topologically non trivial one dimensional (spatial) exicitations exist, which in terms of σ look as kinks (half space with $\sigma = -1$, half space with $\sigma = 1$) and are highly non local. In terms of σ^* a kink is a configuration with a single spin up. When described in terms of σ^* (dual variable) the system is again an Ising model: it is self dual.

The only change is that the Boltzman factors, β and β^*, of the two descriptions are related by

$$\sinh \frac{2}{\beta} = \frac{1}{\sinh \frac{2}{\beta^*}} \tag{1}$$

Ordered region of one description correspond to disordered region of the other and viceversa. In the disordered phase $\langle \sigma \rangle = 0$, $\langle \sigma^* \rangle \neq 0$. $\langle \sigma^* \rangle$ is the disorder parameter.

A similar situation occurs in the 3d XY model[5] (liquid He_4) where 2d configurations (vortices) with a conserved quantum number exist. In this case the dual system is a Coulomb gas[6] and there is no self duality.

In the 4d compact $U(1)$ gauge theory the 3d configurations with topology are monopoles, which condense in the disordered phase producing confinement

of electric charges[7].

Not always in these systems it is possible to explicitely write the partition function in terms of the dual variables.

An alternative approach is to write the disorder operator in terms of the original fields[8] (not their dual). This is particularly practical in numerical simulations. A (non local) operator is written which creates a topological excitation, and therefore carries non zero topological charge, and its vev is measured, in order to detect spontaneous breaking of the dual symmetry[5, 9].

In the case of QCD there exists no clear idea of what the dual description is, except that it is presumably a gauge theory, with a gauge group independent of the colour group, and an interchange of roles between electric and magnetic charges[7, 10].

The topological excitations which condense should have magnetic charge. This supports the idea that confinement is produced by dual Meissner effect in a dual superconducting vacuum[11].

Monopoles exist in QCD in connection with any operator Φ in the adjoint representation: they are exposed by a procedure known as abelian projection[12].

A disorder parameter can be defined, wich detects condensation of the monopoles for each operator Φ. Numerical simulations do demonstrate that monopoles condense inthe vacuum, independent of the choice for Φ[13, 14].

In this paper we want to study the 3d ferromagnet Heisenberg model in a dual way. The motivation for that, besides its intrinsic interest, is twofold

1) Have another example of duality and a check of the construction of the disorder parameter.

2) Have some insight into the abelian projection.

The model presents topological configurations in 2 dimensions, the well known instantons of the 2d $O(3)$ σ model[15, 16].

We will numerically check that the disordered (high temperature phase) of the model is a condensate of such excitations, by use of a non local disorder parameter.

The idea will prove correct and as a check finite size scaling analysis at the Curie point will provide a determination of the critical indices and of the transition temperature, which agree with the ones produced by other methods.

To make connection with gauge theory we shall reformulate the model in terms of a fiber bundle. We shall also consider a gauged version of it, which is a $2+1$ dimensional Georgi-Glashow model. The Heisenberg model can be viewed as the limit of zero gauge coupling of it. The topological charge turns out to be the corresponding limit of t'Hooft's magnetic charge.

The paper is organized as follows. In sect.2 we define the model, and formulate it in terms of a fiber bundle. In sect.3 we define the disorder parameter, and present its numerical evaluation, together with the determination of the transition point and of critical indices. Some remarks on the construction of the disorder parameter are contained in sect.4. Sect.5 describes the gauged version

and discusses the limit of zero gauge coupling. Sect.6 contains some concluding remarks.

2 The model.

The partition function is

$$Z[\beta] = \int \prod_x [d\Omega(x)] \, \exp(-S) \tag{2}$$

with $d\Omega(x)$ the area element on the unit sphere in colour space and

$$S = \frac{1}{2}\beta \sum_{\mu=0}^{2} \sum_x (\Delta_\mu \vec{n}(x))^2 \qquad \Delta_\mu \vec{n}(x) = [\vec{n}(x+\hat{\mu}) - \vec{n}(x)] \tag{3}$$

A generic x dependent $O(3)$ transformation

$$\vec{n}(x) \rightarrow U(x)\,\vec{n}(x) \tag{4}$$

leaves $Z[\beta]$ invariant, in spite of the fact that S is not invariant, since it can be reabsorbed in a change of the variable $\Omega(x)$ which leaves the measure $d\Omega(x)$ invariant.

The continuum version of the model is the nonlinear $O(3)$ $\sigma-$model[1]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\sigma})^2 \qquad (\vec{\sigma}^2 = 1) \tag{5}$$

A gauged version of the model is in the continuum

$$\mathcal{L} = \frac{1}{2}(D_\mu \vec{\sigma})^2 - \frac{1}{4}\vec{G}_{\mu\nu} \cdot \vec{G}_{\mu\nu} \tag{6}$$

$$D_\mu = \partial_\mu - ig\vec{T} \cdot \vec{A}_\mu \qquad \vec{G}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \wedge \vec{A}_\nu$$

which is a 2+1 dimensional Georgi-Glashow model, with the length of the Higgs field frozen to 1.

The Heisenberg model can be considered as the limit $g \rightarrow 0$ of the system (6).

Usually an x independent frame is used in colour space, or 3 fixed unit vectors $\vec{\xi}_i^0$ $(i = 1, 2, 3)$ with

$$\vec{\xi}_i^0 \cdot \vec{\xi}_j^0 = \delta_{ij} \qquad \vec{\xi}_i^0 \wedge \vec{\xi}_j^0 = \vec{\xi}_k^0 \tag{7}$$

However a body fixed frame (BFF) can be used[17], with unit vectors $\vec{\xi}_i(x)$ $(i = 1, 2, 3)$ again obeying

$$\vec{\xi}_i \cdot \vec{\xi}_j = \delta_{ij} \qquad \vec{\xi}_i \wedge \vec{\xi}_j = \vec{\xi}_k \tag{8}$$

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and $\vec{\xi}_3(x) = \vec{n}(x)$.

The frame is determined modulo an x dependent arbitrary rotation around $\vec{n}(x)$.

Since $\vec{\xi}_i^2 = 1$

$$\partial_\mu \vec{\xi}_i(x) = \vec{\omega}_\mu \wedge \vec{\xi}_i(x) \tag{9}$$

or

$$D_\mu \vec{\xi}_i(x) = 0 \tag{10}$$

where we have defined the covariant derivative

$$D_\mu \equiv \partial_\mu - \vec{\omega}_\mu \wedge = \partial_\mu - i\vec{T} \cdot \vec{\omega}_\mu \tag{11}$$

Eq.(10) also implies that

$$[D_\mu(\omega), D_\nu(\omega)] \vec{\xi}_i(x) = 0 \tag{12}$$

or, by the completeness of $\vec{\xi}_i$

$$[D_\mu(\omega), D_\nu(\omega)] = \vec{T} \cdot \vec{G}_{\mu\nu} = 0 \tag{13}$$

$\vec{\omega}_\mu$ is a pure gauge

$$\partial_\mu \vec{\omega}_\nu - \partial_\nu \vec{\omega}_\mu + \vec{\omega}_\mu \wedge \vec{\omega}_\nu = 0 \tag{14}$$

The geometrical meaning of the above analysis is nothing but the very definition of parallel transport. Eq.(14) is true, apart from singularities. A consequence of (14) is that any field configuration $\vec{n}(x)$ can be written as a parallel transport from infinity to the point x of the value $\vec{n}_0$ which the field has at some point at infinity

$$\vec{n}(x) = \mathcal{P} \left(\exp i \int_{\infty, \mathcal{C}} \vec{T} \cdot \vec{\omega}_\mu(x) \, dx^\mu \right) \vec{n}_0 \equiv R(x)\vec{n}_0 \tag{15}$$

$\mathcal{P}$ means path-ordering. The choice of the path $\mathcal{C}$ is irrelevant as long as the field $\vec{\omega}_\mu$ is a pure gauge, or eq.(14) is satisfied. We will show that this is not true, because singularities of the matrix R in eq.(15), which make the connection of the bundle non trivial.

We now analyse the nature of such singularities.

The model has a Noëther current

$$\vec{J}^\mu_{(N)} = \vec{n} \wedge \partial^\mu \vec{n} \tag{16}$$

corresponding to $O(3)$ invariance which is conserved by virtue of the equation of motion

$$\partial_\mu J^\mu_{(N)} = 0 \tag{17}$$

There is, however another current which is identically conserved

$$\vec{J}_\mu = \frac{1}{8\pi} \varepsilon_{\mu\alpha\beta} \partial_\alpha \vec{n} \wedge \partial_\beta \vec{n} \tag{18}$$

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$\vec{J}_\mu$ is parallel to $\vec{n}$, since both $\partial_\alpha \vec{n}$ and $\partial_\beta \vec{n}$ are orthogonal to it.

$$\partial^\mu \vec{J}_\mu = 0 \qquad (19)$$

Also the colour invariant current

$$J_\mu = \vec{n} \cdot \vec{J}_\mu = \frac{1}{8\pi} \varepsilon_{\mu\alpha\beta} \vec{n} \cdot \partial_\alpha \vec{n} \wedge \partial_\beta \vec{n} \qquad (20)$$

is identically conserved

$$\partial^\mu J_\mu = 0 \qquad (21)$$

The conservation law (21) is a consequence of the invariance with respect the x dependent rotations around $\vec{n}(x)$. The action (5) can indeed be written as

$$S = \frac{\beta}{2} (\vec{\omega}_\mu \wedge \vec{n})^2 = \frac{\beta}{2} (\vec{\omega}_\mu^\perp)^2$$

Any rotation around $\vec{n}$ corresponds to an $\vec{\omega}$ parallel to $\vec{n}$, and leaves S invariant. In defining the BFF this invariance reflects in the arbitrariness by a rotation in the choice of $\vec{\xi}_1$, $\vec{\xi}_2$, being $\vec{\xi}_3$ parallel to $\vec{n}$.

The conserved charge corresponding to (21) is

$$\begin{aligned} Q &= \frac{1}{4\pi} \int \vec{n} \cdot (\partial_1 \vec{n} \wedge \partial_2 \vec{n}) d^2 x \qquad (22) \\ &= \frac{1}{4\pi} \int \vec{n} \cdot (\vec{\omega}_1 \wedge \vec{\omega}_2) \, d^2 x \end{aligned}$$

Q is nothing but the Chern number of the $2d$ $O(3)$ $\sigma-$model, which assumes positive or negative integer values,

$$Q = \pm n \qquad (23)$$

On the other hand, by use of eq.(14)

$$\begin{aligned} Q &= \frac{1}{4\pi} \int \vec{n} \cdot (\vec{\omega}_1 \wedge \vec{\omega}_2) \, d^2 x = \frac{1}{4\pi} \int (\partial_1 \vec{\omega}_2 \cdot \vec{n} - \partial_2 \vec{\omega}_1 \cdot \vec{n}) d^2 x = \\ &= \frac{1}{4\pi} \int (\partial_1 (\vec{\omega}_2 \cdot \vec{n}) - \partial_2 (\vec{\omega}_1 \cdot \vec{n})) d^2 x + 2 \frac{1}{4\pi} \int \vec{n} \cdot (\vec{\omega}_1 \wedge \vec{\omega}_2) \, d^2 x \end{aligned}$$

and hence, since the last term equals $2Q$,

$$\pm n = \oint \vec{\omega}_i \cdot \vec{n} dx^i \qquad (24)$$

showing that the field $\vec{\omega}_\mu$ can have a nontrivial connection. There exist singularities of $\vec{\omega}_\mu$ where eq.(14) is not valid.

This can be seen by expressing $\vec{\xi}_i$ in polar coordinates with respect to $\vec{\xi}_i^0$, the fixed frame axes. Then one can compute $\vec{\omega}_\mu$ getting[17]

$$\vec{\omega}_\mu = \begin{pmatrix} \sin\theta \partial_\mu \psi \\ -\partial_\mu \theta \\ -\cos\theta \partial_\mu \psi \end{pmatrix} \qquad (25)$$

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Singularities can occur when $\cos\theta = \pm 1$, and ψ is not defined. The singularity is then of the form

$$\vec{\omega}_\mu^{sing} = \begin{pmatrix} 0 \\ 0 \\ \pm\partial_\mu\psi^{sing} \end{pmatrix}$$

The corresponding field is an abelian field parallel to $\vec{n}$ in the sites where $\vec{n} = \vec{n}_0$

$$\vec{F}_{\mu\nu}^{sing} = \pm\vec{n}_0(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\psi^{sing}$$

The singular field can be explicitly computed for a time independent soliton solution (a 2d instanton independent of x_0), sitting at the origin, getting (see sect.5)

$$\partial_1\omega_2^{sing} - \partial_2\omega_1^{sing} = 2\pi\delta^{(2)}(\vec{x})$$

This is nothing but a Dirac string.

3   The disorder parameter

We will show that solitons condense in the disordered phase of the model ($\beta < \beta_c$), i.e. that the $U(1)$ symmetry eq.(21) is spontaneously broken in the disordered phase. To do that we define a disorder parameter which is the vacuum expectation value (vev) of the creation operator of a soliton.

We start by defining the creation operator of a soliton.

Let $R_q(\vec{x}, \vec{y})$ be a singular time independent rotation which creates a soliton of topological charge q at site $\vec{y}$. We will give and discuss the explicit form of R_q below.

We define the lattice creation operator of a soliton at site $\vec{y}$ and time t as follows

$$\mu_q(\vec{y}, t) = \exp\left\{-\beta\left[\sum_{\vec{x}}(R_q^{-1}(\vec{x}, \vec{y})\vec{n}(\vec{x}, t+1) - \vec{n}(\vec{x}, t))^2 \right. \right. \tag{26}$$
$$\left. \left. -(\vec{n}(\vec{x}, t+1) - \vec{n}(\vec{x}, t))^2\right]\right\}$$

When computing a correlation function of μ's the definition amounts to replace at the time t the time derivative term of the action $[\Delta_0\vec{n}(\vec{x}, t)]^2$ (the second term in parenthesis of eq.(26)) by the first term.

We will compute the correlator

$$\mathcal{D}(x_0) = \langle\mu_{-q}(\vec{0}, x^0)\mu_q(\vec{0}, 0)\rangle \tag{27}$$

i.e. the propagation of a soliton sitting in the origin from time 0 to time x_0.

At large values of $|x_0|$, by cluster property one expects

$$\mathcal{D}(x_0) \simeq A\exp(-M|x_0|) + \langle\mu_q\rangle^2 \tag{28}$$

$\langle\mu_q\rangle \neq 0$ in the thermodinamic limit signals spontaneous breaking of the $U(1)$ symmetry (21).

Our guess is that this spontaneous breaking is related to the phase transition, and that $\langle \mu_q \rangle$ is the disorder parameter of the system, dual to the magnetization in the low temperature phase.

To check that we will compute $\langle \mu_q \rangle$ numerically, and in particular we will study its behaviour around β_c. As a byproduct we shall determine by a finite size scaling analysis β_c and the critical index ν of the spin correlation length, and β_c.

To show that μ_q actually creates a soliton let us compute $\mathcal{D}(x_0)$ of eq.(27) by use of the definition (26) of μ_q

$$\mathcal{D}(x_0) = \frac{\int d\Omega \exp(-\beta S)\mu_{-q}(\vec{0}, x_0)\mu_q(\vec{0}, 0)}{\int d\Omega \exp(-\beta S)} = \frac{Z[S + \Delta S]}{Z[S]} \qquad (29)$$

According to the definition (26) $S + \Delta S$ is obtained from S (eq.(2)) by the replacements at the two time slices $t = 0$ and $t = x_0$

$$t = 0 \qquad (\Delta_0 \vec{n}(\vec{x}, 0))^2 \rightarrow \left(R_q^{-1} \vec{n}(\vec{x}, 1) - \vec{n}(\vec{x}, 0) \right)^2 \qquad (30)$$

$$t = x_0 \qquad (\Delta_0 \vec{n}(\vec{x}, x_0))^2 \rightarrow \left(R_{-q}^{-1} \vec{n}(\vec{x}, x_0 + 1) - \vec{n}(\vec{x}, x_0) \right)^2 \qquad (31)$$

$Z[S + \Delta S]$ can be computed by the change of variables $\vec{n}(\vec{x}, 1) \rightarrow R_q \vec{n}(\vec{x}, 1)$. As observed at the beginning of sect.2 this change leaves the measure invariant. The term $(\Delta_o \vec{n})^2$ is restored to the primitive form, but

$$(\Delta_i \vec{n})^2 \rightarrow (\Delta_i R_q \vec{n})^2$$

(i.e. a soliton has been created at time $t = 1$) and

$$\Delta_0 \vec{n}(\vec{x}, 2) \rightarrow (R_q^{-1} \vec{n}(\vec{x}, 2) - \vec{n}(\vec{x}, 1))$$

which translates the change (30) at $t = 1$.

The construction can be repeated until time x_0 is reached, when the construction produces again a soliton and

$$(\Delta_0 \vec{n}(\vec{x}, x_0))^2 \rightarrow \left(R_{-q}^{-1} \vec{n}(\vec{x}, x_0 + 1) - R_q \vec{n}(\vec{x}, x_0) \right)^2 = (\Delta_0 \vec{n}(\vec{x}, x_0))^2$$

since $R_{-q}^{-1} = R_q$. This shows that $\mathcal{D}_q(x_0)$ actually describes a soliton propagating from $t = 0$ to $t = x_0$.

Instead of $\langle \mu \rangle$, the quantity $\rho = \frac{d}{d\beta} \ln \langle \mu \rangle$ will be measured[5]. Since $\langle \mu \rangle_{\beta=0} = 1$
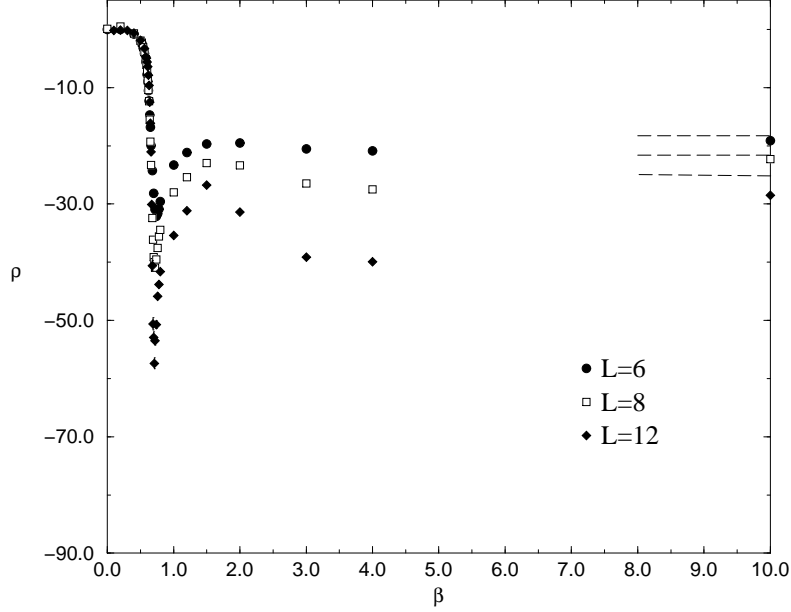
$$\mu = \exp \left[\int_0^\beta \rho(x) dx \right] \qquad (32)$$

$\rho(\beta)$ is shown in fig.1 for different sizes of the lattice . At high β's the integral defining $\mathcal{D}(x_0)$ is gaussian and ρ can be explicitly computed. The result is

$$\rho = -c_1 L + c_2 \qquad (\beta > \beta_c) \qquad (33)$$

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with $c_1 > 0$. The comparison of this analytic computation to numerical results is displayed in the figure.

Fig.1 ρ vs β. The perturbative evaluation at high β is shown by the dotted lines for comparison.

$\langle\mu\rangle$ is an analytic function of β for any finite L, and therefore it cannot vanish in a region of β's without vanishing identically everywhere. Only in the thermodynamical limit Lee-Yang singularities are produced and $\langle\mu\rangle$ can be identically zero for $\beta > \beta_c$, as a honest disorder parameter.

At $\beta < \beta_c$, ρ tends to a finite limit as $V \to \infty$ (fig.2), or $\langle\mu\rangle$ tends to a value different from zero, implying that $U(1)$ symmetry, eq.(21), is spontaneously broken.

Around β_c a finite size analysis can be made to determine β_c and the critical index ν. The argument is that, for dimensional reasons[6].

$$\langle\mu\rangle = f(\frac{L}{\xi}, \frac{a}{\xi}) \tag{34}$$

with ξ the correlation length,a the lattice spacing, L the lattice size. Near the critical point ξ diverges with a critical index ν

$$\xi \simeq |\beta_c - \beta|^{-\nu} \tag{35}$$

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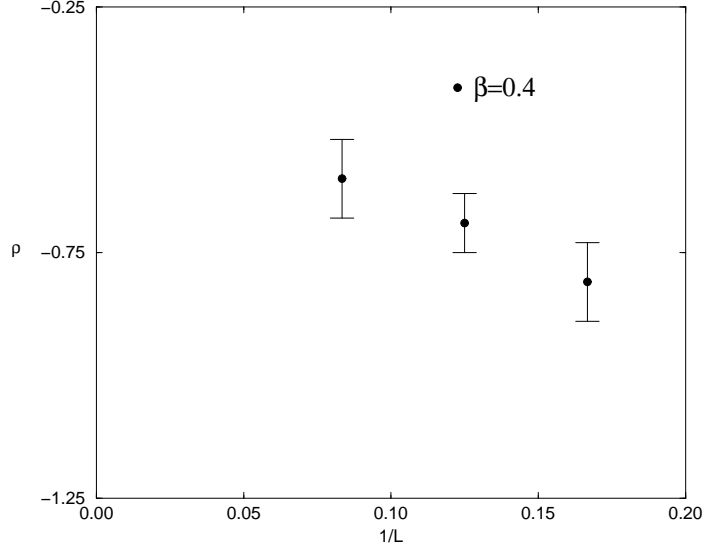
Fig.2 ρ at high temperature as a function of $1/L$, (L size of the lattice).

As ξ goes large, the dependence of μ on a/ξ can be neglected, and

$$\langle\mu\rangle = f(\frac{L}{\xi}, 0) = \Phi(L^{1/\nu}(\beta_c - \beta)) \tag{36}$$

This induces on $\rho = d\ln\langle\mu\rangle/d\beta$ the scaling law

$$\frac{\rho}{L^{1/\nu}} = F(L^{1/\nu}(\beta_c - \beta)) \tag{37}$$

Corresponding to the correct value of ν and β_c determinations of $\rho L^{-1/\nu}$ coming from lattices of different size should follow the same curve if plotted versus $L^{1/\nu}(\beta_c - \beta)$. The quality of scaling is shown in fig.3. A best fit procedure gives

$$\nu = 0.70 \pm 0.02 \; [0.704(6)] \qquad \beta_c = 0.695 \pm 0.003 \; [0.6929(1)] \tag{38}$$

For comparison the official values[15, 18, 19] are shown in square brackets.

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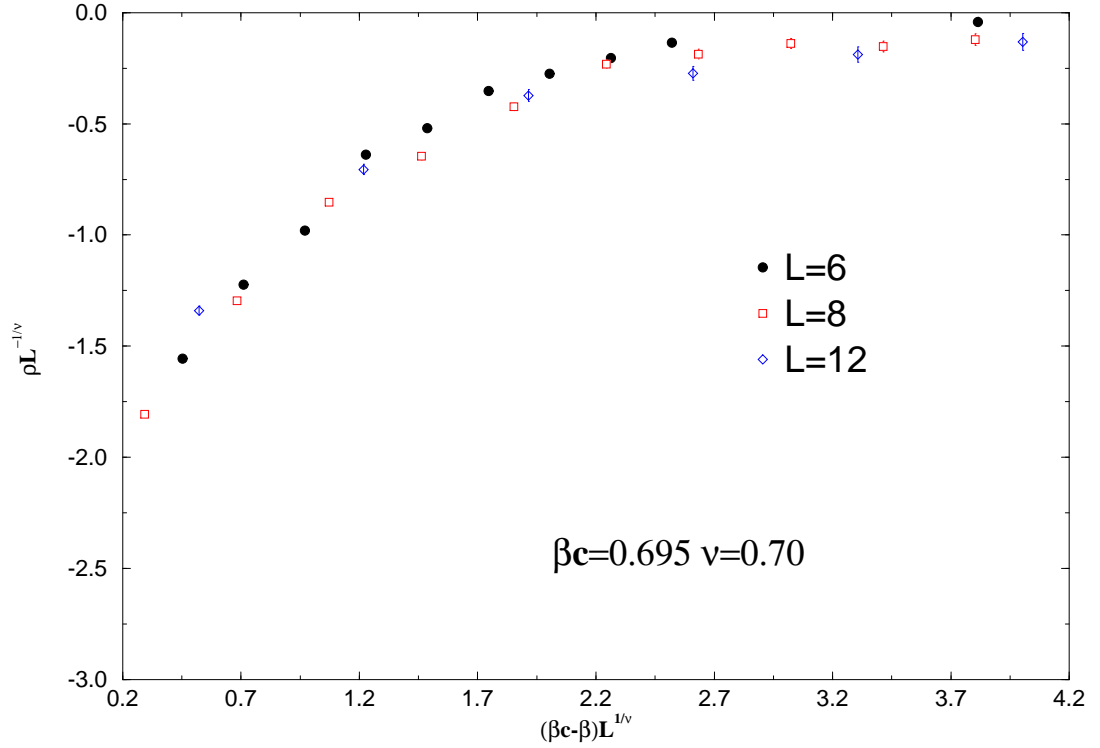
Fig.3 Finite size scaling at the optimal values of β_c and ν.

4   On the construction of $R_q(x, y)$

We want now to analyse in detail the definition and the properties of the rotation R_q, introduced in the previous section (eq.(26)). In principle the model is defined with the boundary condition that

$$\vec{n}(x) \rightarrow_{|x| \to \infty} \vec{n}_0$$

to have a finite action. A time independent rotation $R_q(x, y)$ which creates a vortex when acting on a uniform configuration in space

$$\vec{n}(\vec{x}, t) = \vec{n}_0 = (0, 0, 1)$$

is known

$$R_q(x, y) = R_z(\theta) R_x(f) R_z^{-1}(\theta) \tag{39}$$

with

$$\theta = arctg \frac{(\vec{x} - \vec{y})_2}{(\vec{x} - \vec{y})_1} \qquad f = f(r), \quad r = \sqrt{(\vec{x} - \vec{y})^2} \tag{40}$$

and with the boundary conditions

$$f(\infty) = 0 \qquad f(0) = \pi \tag{41}$$

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The connection $\vec{\omega}_\mu$ produced by R_q

$$\vec{\omega}_\mu = \partial_\mu R_q \, R_q^{-1}$$

is, in polar coordinates

$$\vec{\omega}_0 = 0 \qquad \vec{\omega}_\theta = \begin{pmatrix} \sin\theta\sin f \\ \cos\theta\sin f \\ 1-\cos f \end{pmatrix} \qquad \vec{\omega}_r = -f' \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} \qquad (42)$$

The topological charge, as given by eq.(22), is

$$\frac{1}{4\pi} \oint_{C_1} \vec{\omega}_\theta \cdot \vec{n} d\theta + \frac{1}{4\pi} \oint_{C_2} \vec{\omega}_\theta \cdot \vec{n} d\theta = (1-\cos f(0))n_3(\vec{y}) = 1 \qquad (43)$$

where the line integral is taken on a small circle C_1 around the point $\vec{y}$ and a circle C_2 at $|\vec{x}| = \infty$. The first integral gives $(1-\cos f(0))n_3(\vec{y})$, the second $(1-\cos f(\infty))n_3(\vec{\infty})$. we have chosen $f(0) = \pi$ $f(\infty) = 0$. Alternatively one can take $f(\infty) = \pi$ and $f(0) = 0$, and then the role of the two circles C_1 and C_2 is interchanged. If we act on a generic configuration $\vec{n}(\vec{x})$

$$\vec{n}(x) \rightarrow R_q\vec{n}(x) = \vec{n}'$$

then we can compute the variation δQ of the topological charge produced by R_q

$$
\begin{aligned}
\delta Q &= \frac{1}{4\pi} \int \vec{n}'(x) \cdot (\partial_1\vec{n}' \wedge \partial_2\vec{n}') \, d^2x - \frac{1}{4\pi} \int \vec{n}(x) \cdot (\partial_1\vec{n} \wedge \partial_2\vec{n}) \, d^2x \\
&= \frac{1}{4\pi} \int \vec{n} \cdot (\partial_1 R_q R_q^{-1}\vec{n} \wedge \partial_2\vec{n}) d^2x + \frac{1}{4\pi} \int \vec{n} \cdot (\partial_1\vec{n} \wedge \partial_2 R_q R_q^{-1} + \\
&+ \frac{1}{4\pi} \int \vec{n} \cdot (\partial_1 R_q R_q^{-1}\vec{n} \wedge \partial_2 R_q R_q^{-1}\vec{n}) d^2x \qquad (44)
\end{aligned}
$$

Assuming the form of eq.(39) for R_q the result is

$$\delta Q = \frac{1}{2} \left[\vec{n}(\vec{y})\vec{\omega}_\theta(\vec{y}) - \vec{n}(\infty)\vec{\omega}_\theta(\infty) \right] \qquad (45)$$

with $\vec{\omega}_\theta$ defined in eq.(42). If $f(\infty) = \pi$, $f(0) = 0$ $\vec{\omega}_\theta(0) = 0$, $\vec{\omega}_\theta(\infty) = (0,0,2)$ and $\delta Q = 1$. The rotation R_q changes by 1 the number of vortices. In our numerical simulations we have first tried a fixed b.c., $\vec{n}_b = \vec{n}_0 = (0,0,1)$. Since, however, the correlation length tends to be large when approaching the critical point, the lattice tends to be dominated by surface effects with very little bulk of the system, for reasonable lattices sizes. An alternative possibility is to have one single line parallel to the time axis $(\vec{x} = \vec{y})$ on which $\vec{n} = \vec{n}_0$ and put the vortex generated by R_q at the site $\vec{y}$. This strongly reduces the boundary effect and makes the lattice much bigger. A third possibility is to use plain periodic b.c. Then in principle eq.(45) as it stands from the continuum, only gives the correct δQ if by chance $\vec{n}(\vec{y}) = \vec{n}_0$. More than that R_q is defined only if that is

true. However if we change the configuration by putting $\vec{n}(\vec{y}) = \vec{n}_0$ the action only changes by $\mathcal{O}(1/L^2)$, the sum in eq.(44) is not affected by that change to $\mathcal{O}(1/L^2)$. Indeed for small values of correlation length the different boundary conditions described above give the same result within errors.

We finally compute the connection $\vec{\omega}_\mu$ and the gauge field $G_{\mu\nu}(\omega)$ for the configuration of a vortex. We shall refer to a configuration

$$n_+ = \frac{2w}{1 + |w|^2|} \qquad n_3 = \frac{1 - |w|^2}{1 + |w|^2} \tag{46}$$

with w a meromorfic funtion of degree q $w = \prod_1^q \frac{z - a_i}{z - b_i}$. Let us put $\vec{\omega}_\mu = \vec{\omega}_\mu^\perp + f_\mu \vec{n}$. We get

$$\begin{aligned}
G_{\mu\nu}(\omega) &= \vec{G}_{\mu\nu}^\perp + \vec{G}_{\mu\nu}^L \tag{47}\\
\vec{G}_{\mu\nu}^\perp &= \partial_\mu \vec{\omega}_\nu^\perp - \partial_\nu \vec{\omega}_\mu^\perp - \vec{\omega}_\mu^\perp \wedge \vec{\omega}_\nu^\perp \\
\vec{G}_{\mu\nu}^L &= (\partial_\mu f_\nu - \partial_\nu f_\mu)\vec{n}
\end{aligned}$$

Since only $\vec{\omega}_\mu^\perp$ is defined, we will determine f_μ in such a way that $G_{\mu\nu}(\omega) = 0$. The solution is

$$f_1 + if_2 = in_3 \frac{1}{\bar{w}} \frac{\partial \bar{w}}{\partial \bar{z}} \tag{48}$$

The topological charge according to eq.(43) is then

$$Q = \frac{1}{4\pi} \oint_\Gamma f_i dx^i = q \tag{49}$$

Γ is the contour sum of circles around the singularities of f, i.e. around the poles and zeros of w.

5 The gauge version.

The gauge version is defined by eq.(6) in the continuum. It is a 2+1 dimensional Georgi-Glashow model with the size of the Higgs field frozen. On the lattice it becomes

$$S = \beta \sum_{x,\mu} [D_\mu \vec{n}(x)]^2 + \beta' \text{Tr} \{\Pi_{\mu\nu}\} \tag{50}$$

where

$$D_\mu \vec{n}(x) = \vec{n}(x + \hat{\mu}) - U_\mu(x)\vec{n}(x) \tag{51}$$

and $\Pi_{\mu\nu}$ is the plaquette. Again a body fixed frame can be defined, and with it $\vec{\omega}_\mu$, exactly as in sect.2. Now

$$D_\mu \vec{n} = \partial_\mu \vec{n} + (-g\vec{A}_\mu + \vec{\omega}_\mu) \wedge \vec{n} \tag{52}$$

while $\vec{\omega}_\mu$ and $g\vec{A}_\mu$ transform both as gauge fields. Under infinitesimal gauge transformation both $\vec{\omega}_\mu$ and $\vec{A}_\mu$ transform as gauge fields

$$
\begin{aligned}
g\vec{A}_\mu &\rightarrow \vec{\delta} \wedge g\vec{A}_\mu + \partial_\mu \vec{\delta} \\
\vec{\omega}_\mu &\rightarrow \vec{\delta} \wedge \vec{\omega}_\mu + \partial_\mu \vec{\delta}
\end{aligned}
\tag{53}
$$

The combination $\vec{\omega}_\mu - g\vec{A}_\mu$ is covariant[17]. In the absence of singularities the field $\vec{\omega}_\mu$ can be gauged away by a transformation which brings $\vec{n}(x)$ to $\vec{n}_0$, $R^{-1}(x)$, where R is defined by $\vec{n}(x) = R(x)\vec{n}_0$. If the transformation is singular, $\vec{G}_{\mu\nu}(x)$ is not covariant but acquires a singular additive term, parallel to $\vec{n}$, $\vec{G}_{\mu\nu}(\omega)$

$$
\vec{G}_{\mu\nu} \rightarrow \vec{G}_{\mu\nu}(\vec{A}) + \vec{G}_{\mu\nu}(\vec{\omega})
\tag{54}
$$

The gauge transformation $R^{-1}(x)$ is called an abelian projection. Going to the BFF makes $\vec{\omega}_\mu = 0$ and $\partial_\mu \vec{n} = 0$. The $U(1)$ invariance corresponding to the rotations around $\vec{n}$ becomes in this frame a $U(1)$ abelian gauge invariance. The corresponding gauge field is $\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$, or

$$
F_{\mu\nu} = \vec{n} \cdot \vec{G}_{\mu\nu} - g(\vec{A}_\mu \wedge \vec{A}_\nu) \cdot \vec{n}
\tag{55}
$$

By use of eq.(55)

$$
g(\vec{A}_\mu \wedge \vec{A}_\nu)_3 = \frac{1}{g}\vec{n} \cdot (D_\mu \vec{n} \wedge D_\nu \vec{n})
$$

and

$$
F_{\mu\nu} = \vec{n} \cdot \vec{G}_{\mu\nu} - \frac{1}{g}\vec{n} \cdot (D_\mu \vec{n} \wedge D_\nu \vec{n})
\tag{56}
$$

which is the 't Hooft tensor, and is gauge invariant. Calling

$$
\begin{aligned}
j_\mu &= \partial^\mu F_{\mu\nu} \\
j_\mu^* &= \varepsilon_{\mu\alpha\beta} F^{\alpha\beta}
\end{aligned}
\tag{57}
\tag{58}
$$

j_μ^* is identically conserved, since in the abelian projected frame $F_{\mu\nu}$ is a curl. In the limit $g \rightarrow 0$, gj_μ^* coincides with the current (20). The above analysis shows that the Heisenberg model can be seen as the $g \rightarrow 0$ limit of a Higgs model. The abelian projection is the transformation to BFF. Its singularities depend on the Higgs field configurations: the gauge field for the connection $\vec{\omega}_\mu$ is present both in the gauged and in the simple version of the model.

6 Concluding remarks

A disorder parameter $\langle \mu \rangle$ has been defined for the phase transition of demagnetization in the 3d Heisenberg model. $\langle \mu \rangle$ vanishes in the magnetized phase, and is non zero in the disordered phase, signalling condensation of vortices in the vacuum. The vortices are the instantons of the 2d version of the model. The critical index ν and the transition temperature β_c can be determined by

a finite size scaling analysis, and they agree with the values obtained from the side of ordered phase. Duality implies a non trivial topological structure of the model. Vortices can be viewed as gauge singularties resulting from the abelian projection. In fact they have a physical role on the dynamics of the system.

References

[1] H.A. Kramers, G.H. Wannier, *Phys. Rev.* **60**, 252 (1941).

[2] L.P. Kadanoff, H.Ceva, *Phys. Rev.* B **3** 3918 (1971).

[3] G. 't Hooft, *Nucl. Phys.* **B138**, 1, (1978).

[4] N. Seiberg, E. Witten, *Nucl. Phys.* **B426**, 19, 1994, erratum *ibidem* **B430**, 485, 1994.

[5] G. Di Cecio, A. Di Giacomo, G. Paffuti, M. Trigiante, *Nucl. Phys.* **B489**, 739, (1997).

[6] G. Kohring, R.E. Shrock, P. Wills, *Phys. Rev. Lett.* **57**, 397, (1986).

[7] T. Banks, R. Myerson, J. Kogut, *Nucl. Phys.* **B129**, 493, 1977

[8] E. Marino, B. Schroer, J.A. Swieca, *Nucl. Phys.* **B200**,473,(1982).

[9] A. Di Giacomo, G. Paffuti, *Phys. Rev.* **D56**, 6816, (1997).

[10] H. M. Chan, J. Feridan, S.T. Tsou, *Phys. Rev.* **52**, 6134, (1995)

[11] G. 't Hooft, in "High Energy Physics", EPS International Conference, Palermo 1975, ed. A. Zichichi; S. Mandelstam, *Phys. Rep.* **23C**, 245 (1976).

[12] G. 't Hooft, *Nucl. Phys.***B190**,455, (1981).

[13] L. Del Debbio, A. Di Giacomo, G. Paffuti, P. Pieri, *Phys. Lett.* **B355**,255, (1995).

[14] A. Di Giacomo, B.Lucini, L.Montesi, G. Paffuti, *Nucl. Phys.* **B** (Proc. Suppl.) **63** (1998) 540, and in preparation.

[15] E. Brezin, J. Zinn-Justin, *Nucl. Phys.* **B257**, 867, (1985).

[16] A.A. Belavin, A.M. Polyakov, *JETP Lett.*, **22**, 245, (1975).

[17] A. Di Giacomo, M. Mathur, *Phys. Lett.* **B400**, 129, (1997).

[18] P. Peczak, A.L. Ferrenberg, D.P. Landau, *Phys. Rev.* **43**, 6087, (1991).

[19] C. Holm, W. Janke, *Phys. Rev.* **B48**, 936, (1993).

[20] G. 't Hooft, *Nucl. Phys.* **B79**, 276, (1974).